ECE 874 - Spring 2012 **Test 2 In-Class Component** Name ______________________

(80 points total)

1. (10pts) Describe a control design approach suitable for achieving tracking control for each systems:

| | |
|---|---|
| $\dot{x} = \frac{1}{2}x_1^2 + \frac{1}{2}(x_1 - x_2)^2 + u$ | Based on **exact model knowledge**, use feed-forward cancellation on the destabilizing terms $\frac{1}{2}x_1^2 + \frac{1}{2}(x_1 - x_2)^2$ and the derivative of the desired trajectory and add a stabilizing feedback term, $x_d$–$x$. |
| $\dot{x} = \sin(ax) + u$<br>where $a$ is an unknown constant parameter | The nonlinear term sin(ax) is not linear parameterizable, use a **robust controller** for unknown term. |
| $\ddot{x} = 3\sin(x) + 2u$ | Use the filtered tracking error, $r = \dot{e} + \alpha e$, to reduce the order of the tracking problem then design an **exact model knowledge** feed-forward and stabilizing control. |
| $\ddot{x} = b\sin(x) + u$<br>$b$ is known constant parameter<br>$\dot{x}$ is not measurable | Design **observer** for $\dot{x}$ and then include in controller. Observation error and tracking error must be included in single Lyapunov function. |
| $\ddot{x} = f(x) + u$<br>where $f(x)$ is an unknown function of only $x$ | Use filtered tracking error, $r = \dot{e} + \alpha e$, and design a robust controller. |

2. (10pts) You have come to the point in your analysis where you find:

$$\dot{V} = -k_1 s^2 - xs - k_2 x^2,$$

finish the analysis by showing $\dot{V}$ is negative semi-definite in either $x$ or $s$ (your choice). What are the conditions on $k_1$ and $k_2$?

$$\dot{V} = -k_1 s^2 - xs - k_2 x^2 \le -k_1 s^2 + |x||s| - k_2 x^2$$

$$\dot{V} \le -k_1 s^2 + |x||s| - k_2 x^2 = -k_1 s^2 + |x||s| - (k_2)|x|^2 = -k_1 s^2 + (|s| - k_2|x|)|x|$$

$$\dot{V} \le -k_1 s^2 + (|s|)\frac{|s|}{k_2} = -\left(k_1 - \frac{1}{k_2}\right)s^2$$

Choose $k_1 > \dfrac{1}{k_2}$

A better solution is:

$\dot{V} = -k_1 s^2 - xs - k_2 x^2 \le -k_1 s^2 + |x||s| - k_2 x^2$

let $k_2 = k_{21} + k_{22}$

$\dot{V} \le -k_1 s^2 + |x||s| - k_2 x^2 = -k_1 s^2 + |x||s| - (k_{21} + k_{22})|x|^2 = -k_1 s^2 - k_{21}|x|^2 + \left(|s| - k_{22}|x|\right)|x|$

$\dot{V} \le -k_1 s^2 - k_{21}|x|^2 + \left(|s|\right)\frac{|s|}{k_{22}} = -\left(k_1 - \frac{1}{k_{22}}\right)s^2 - k_{21}|x|^2$

Choose $k_1 > \frac{1}{k_{22}}; k_{21} > 0$

3. (20 pts) Given the following system:

$\dot{x}_1 = 2x_1 + u$

Design a tracking controller $u$ so that the state $x_1$ follows $x_{1d}$.

Assume that the desired trajectory and the first derivative exists and is bounded.

Show $V$ and $\dot{V}$ for your control design and that all signals are bounded.

$\dot{x}_1 = 2x_1 + u$

$e = x_{1d} - x_1$

$\dot{e} = \dot{x}_{1d} - \dot{x} = \dot{x}_{1d} - 2x_1 - u$

$V = \frac{1}{2}e^2$

$\dot{V} = e\dot{e} = e\left(\dot{x}_{1d} - 2x_1 - u\right)$

design $u = \dot{x}_{1d} - 2x_1 + ke$

$\dot{V} = -e^2$

V is PD, radially unbounded, $\dot{V}$ is ND $\Rightarrow$ $e \to 0$

$e \to 0 \Rightarrow x_1 \to x_{1d}$ , $x_{1d}$ bounded $\Rightarrow$ $x_1$ is bounded

$e \to 0$, $x_1$ is bounded, $\dot{x}_{1d}$ bounded $\Rightarrow$ $u$ is bounded

$\dot{x}_1$ is bounded, $u$ is bounded $\Rightarrow$ $\dot{x}_1$ is bounded

4. (10pts) You have started to design a tracking controller, *u(t)*, for the system: $\ddot{x} = a\dot{x} + bx + c + u$
The desired trajectory, $x_d$, and the first two derivatives exist and are bounded.

Starting with $V = \frac{1}{2}e^2 + \frac{1}{2}r^2$

you specified $u = \ddot{x}_d - \alpha\dot{e} - a\dot{x} - bx - c + r + e$

to yield $\dot{V} = -\alpha e^2 - r^2$.

Finish this design by proving that the controller will work and that all signals remain bounded.

$V$ is $PD$ and radially unbounded, $\dot{V}$ is ND $\Rightarrow r \to 0, e \to 0$ $\Rightarrow$ the controller achieves the tracking objective

$e \to 0, x_d$ is bounded $\Rightarrow x$ is bounded

since $e, \mathrm{r} \to 0 \Rightarrow \dot{\mathrm{e}} \to 0$

$\dot{\mathrm{e}} \to 0, \dot{x}_d$ is bounded $\Rightarrow \dot{x}$ is bounded

$\ddot{x}_d$ is bounded, $\dot{\mathrm{e}} \to 0, \dot{x}$ is bounded, $x$ is bounded, $r \to 0, e \to 0,$ c is a bounded constant $\Rightarrow u$ is bounded

$\dot{x}$ is bounded, $x$ is bounded, c is a bounded constant, $u$ is bounded $\Rightarrow \ddot{x}$ is bounded

$\Rightarrow$ all signals remain bounded

5. (20 pts) Design a tracking controller, $u(t)$, for the system:

$$\left(x^2+1\right)\dot{x} = -x + u$$

Assume that the desired trajectory, $x_d$, and the first derivative exists and is bounded.

Just show $V$ and $\dot{V}$ for your design (I will interpret the result from the correct $V$ and $\dot{V}$,

i.e. don't worry about stating result or showing boundedness of signals).

$$\dot{x} = -\frac{x}{\left(x^2+1\right)} + \left(\frac{1}{x^2+1}\right)u$$

$$e = x_d - x$$

$$\dot{e} = \dot{x}_d - \dot{x} = \dot{x}_d + \frac{x}{\left(x^2+1\right)} - \left(\frac{1}{x^2+1}\right)u$$

$$V = \tfrac{1}{2}e^2$$

$$\dot{V} = e\dot{e} = e\left(\dot{x}_d + \frac{x}{\left(x^2+1\right)} - \left(\frac{1}{x^2+1}\right)u\right)$$

design $u = \left(x^2+1\right)\left(\dot{x}_d + \dfrac{x}{\left(x^2+1\right)} + ke\right)$

$$\dot{V} = -e^2$$

6. (10 pts) Fill in the 4 blank steps below.

You are designing an observer for $\dot{x}$ in the system

$$\ddot{x} = a\sin(x) + bx + c\cos(x) + u$$

where $a$, $b$, and $c$ are known constants.

$\tilde{x} = x - \hat{x}$

$s = \dot{\tilde{x}} + \alpha\tilde{x}$ so that $\dot{s} = \ddot{\tilde{x}} + \alpha\dot{\tilde{x}} = \ddot{x} - \ddot{\hat{x}} + \alpha\dot{\tilde{x}}$

Propose $V = \frac{1}{2}\tilde{x}^2 + \frac{1}{2}s^2$

$\dot{V} = \tilde{x}\dot{\tilde{x}} + s\dot{s} = \tilde{x}\dot{\tilde{x}} + s\left(\ddot{x} - \ddot{\hat{x}} + \alpha\dot{\tilde{x}}\right)$

rearrange definition of s: $\dot{\tilde{x}} = s - \alpha\tilde{x}$

$\dot{V} = \tilde{x}\left(s - \alpha\tilde{x}\right) + s\left(\ddot{x} - \ddot{\hat{x}} + \alpha\dot{\tilde{x}}\right)$

$= -\alpha\tilde{x}^2 + s\tilde{x} + s\left(\ddot{x} - \ddot{\hat{x}} + \alpha\dot{\tilde{x}}\right)$

Substitute the open-loop system ($\ddot{x}$):

$\dot{V} = -\alpha\tilde{x}^2 + s\tilde{x} + s\left(a\sin(x) + bx + c\cos(x) + u - \ddot{\hat{x}} + \alpha\dot{\tilde{x}}\right)$

You would like to have $\dot{V}$ be negative definite (ND), design $\ddot{\hat{x}}$ to make this happen:

$\ddot{\hat{x}} = \boxed{a\sin(x) + bx + c\cos(x) + u + \tilde{x} + s + \alpha\dot{\tilde{x}}}$

Which makes

$\dot{V} = \boxed{-\alpha\tilde{x}^2 - s^2}$

$V$ is PD and radially unbounded, $\dot{V}$ is ND

$\Rightarrow \tilde{x}, s \to 0 \Rightarrow \hat{x} \to x$

$\Rightarrow \dot{\tilde{x}} = s - \alpha\tilde{x} \to 0 \Rightarrow \dot{\hat{x}} \to \dot{x}$

$\Rightarrow$ observer is bounded if $x, \dot{x}$ are bounded

Now arrange the observer you designed above into an implementable form.

The two-part implementation of the filter is:

$\dot{\hat{x}} = \boxed{p - (\tilde{x}) - \alpha\tilde{x} = p + (1+\alpha)\tilde{x}}$

$\dot{p} = \boxed{a\sin(x) + bx + c\cos(x) + u - \tilde{x} - \alpha\tilde{x} = a\sin(x) + bx + c\cos(x) + u + (1+\alpha)\tilde{x}}$